

Aspects of Gauge Theory - Gravity Correspondence

S. Randjbar-Daemi

the Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11
Trieste, Italy

Abstract

A brief review of aspects of gravity gauge theory correspondance inspired by string theory is presented. *

1. Introduction

Recent developments in superstring theory have opened up new avenues for closer connections of this theory with 4-dimensional physics. Of particular significance is our better understanding of the physics of extended objects, or branes, which appear as classical solutions in string theories. Studies of D -brane physics in particular have given us the new perspective of regarding open strings as the excitation states of these configurations [1]. This new insight requires a very deep and still not well understood interplay between Yang-Mills and gravitational interactions.

One of the main ideas underlying recent excitements is a conjecture of Maldacena [2], known as the AdS/CFT correspondance, which suggests a new connection between gravitational and Yang-Mills forces. This conjecture states that the strong coupling limits of certain gauge theories, a limit in which the conventional perturbative methods fail, are dual to type IIB string theory on a background geometry of $AdS_5 \times X_5$, where X_5 is an Einstein manifold with a cosmological constant $\Lambda = -\Lambda_{AdS_5}$. The case of $X_5 = S^5$ has been studied extensively. In this case the dual

*Based on invited talks given at the International Symposium on Quantum Theory and Symmetries (8-22 July 1999, Goslar, Germany) and the Sixth International Wigner Symposium (WIGSYM6) (16-22 August 1999, Istanbul, Turkey).

gauge theory is the $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$ with the gauge group $U(N)$. Note that the strong coupling here refers to the large values of the 'tHooft coupling $\lambda = g_{\text{YM}}^2 N$. We are thus dealing with large N gauge theory in which the leading order term corresponds to planar Feynman graphs with spherical topology. The subleading terms correspond to surfaces of higher genus [3].

Stringy excitations in gauge theories have often been assumed to be responsible for confinement of quarks in QCD. This picture becomes more plausible in the lattice formulation of Yang-Mills theory. The existence of stringy solutions in gauge theory supports this idea, too. Another, perhaps stronger, evidence for a stringy regime in gauge theories is obtained in the large N expansion. In this approach the gauge theory Feynman graphs go over to string loop graphs, i.e. in a sense the Feynman perturbation theory of a gauge theory with infinite number of colours gives rise to string perturbation theory. The two dimensional Yang-Mills partition function is known to be equivalent to a quasi topological closed string theory in the limit of large N [4]. On the other hand, it is also well known that all consistent string theories do contain gravity in their perturbative spectrum. It becomes plausible then to speculate that there might be a deep interconnection between gauge theories and the gravitational physics. Infact recent developments give a reason why all the previous attempts at deriving a convincing string picture from a gauge theory lagrangian have failed. One thing which the *AdS/CFT* correspondence have taught us is that the strong coupling description of gauge theory requires infinite number of new degrees of freedom provided by the gravitational modes of the dual theory.

In this contribution, after briefly reviewing some elementary facts about large N gauge theory and the geometry of conifolds, we give a short description of the original *AdS/CFT* correspondence with the maximal supersymmetry. We shall then elaborate on some new developments with lesser amount of supersymmetries. More specifically, we shall discuss D_3 -brane configurations for which the transverse space is a non-compact Calabi-Yau 3-fold which is known as a conifold. Klebanov and Witten have shown that this configuration leads to a $\mathcal{N} = 1$ superconformal theory with the gauge group $SU(N) \times SU(N)$, where N is the number of D_3 -branes [5]. In a sense this theory offers a better test of the Maldacena conjecture than the example of the $AdS_5 \times S_5$ background. In this latter case the maximal symmetry of the background dictates the spectrum and the structure of some of the Green functions. The conifold background, on the other hand, has the minimum amount of conformal supersymmetries and therefore the matching of Yang-Mills states on the boundary and the Kaluza-Klein states in the bulk is not entirely dictated by group theory.

Conifolds are ubiquitous in string theory. They appear in the Calabi-Yau compactifications of string theory [6] and the effective action governing low energy physics of moduli fields [7]. We shall elaborate briefly on these points and some others in section 3.

Conifolds also enter the $c = 1$ non critical string theory [9]. The partition function of this theory happens to be equal to the coefficient of certain terms in the the Calabi-

Yau compactifications of type IIB superstrings near a conifold singularity [8]. One can regard the type IIB string near a conifold point as the infinite r limit of a D_3 brane configuration. Starting from this infinite r configuration we can continuously approach the throat region near $r = 0$, where the D_3 brane geometry factorises into $AdS_5 \times T^{11}$.¹ It is in the background of this throat region that the Klebanov-Witten $\mathcal{N} = \infty$ superconformal gauge theory is dual to the type IIB supergravity. One may then ask about a possible role for the $c = 1$ theory in the boundary Yang-Mills theory which arises in the AdS/CFT correspondence. This point will be touched upon in section [6].

2. Conical Singularity [6]

Consider a Calabi-Yau three folds defined by an algebraic equation

$$F(\xi_1, \dots, \xi_4) = 0$$

A nodal singularity is defined to be a point on the surface at which the first derivatives of F vanish, viz;

$$\frac{\partial F}{\partial \xi_i} = 0 \quad i = 1, \dots, 4$$

Near a nodal point one can approximate the above algebraic equation by a quadratic

$$\xi_1^2 + \dots + \xi_4^2 = 0 \quad \xi_i \in \mathbb{C}$$

The surface defined by this equation will be denoted by Y_6 . Note that if $\vec{\xi} \in Y_6$ then $\lambda \vec{\xi} \in Y_6$ for any $\lambda \in \mathbb{C}$. Thus Y_6 is the space of lines through the origin. We thus have a cone with its apex at $\xi_1 = \dots = \xi_4 = 0$.

It is convenient to introduce real coordinates $\vec{\xi} = \vec{x} + i\vec{y}$ where $\vec{x} \& \vec{y} \in \mathbb{R}^4$. In these coordinates the equation of surface becomes

$$\vec{x}^2 - \vec{y}^2 = 0$$

$$\vec{x} \cdot \vec{y} = 0$$

The base of the cone is obtained by intersecting the surface Y_6 with an S^7 centered at the apex

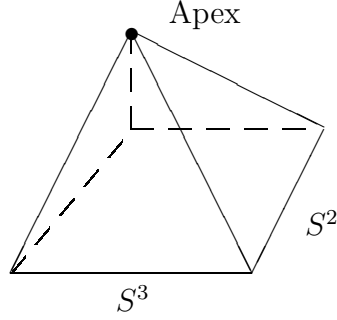
$$\vec{x}^2 + \vec{y}^2 = r^2$$

We thus obtain for the base

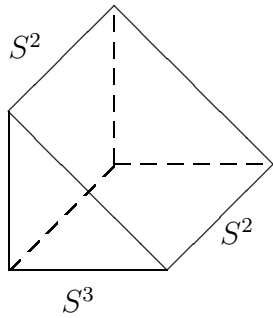
$$\begin{cases} \vec{x}^2 = \vec{y}^2 = \frac{r^2}{2} \\ \vec{x} \cdot \vec{y} = 0 \end{cases}$$

Topologically this is $S^3 \times S^2$,

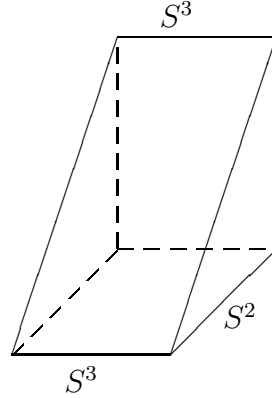
¹ The geometry of T^{11} will be explained in section 5.



The singularity at the apex can be resolved in two different ways by replacing the singular point either with a S^2 (small resolution) or with a S^3 (deformation). Symbolically these two possibilities can be represented as in the following figures:



Small resolution

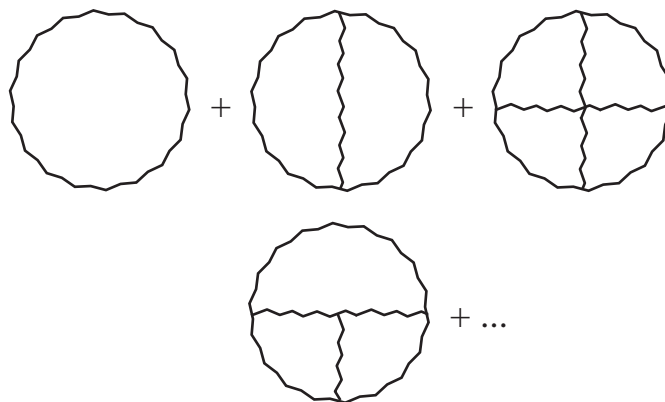


Deformation

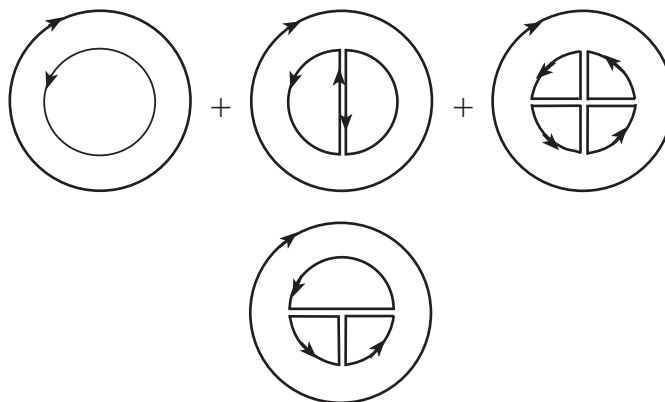
3. Conifolds in String Theory

In standard perturbation theory one expands each amplitude in a power series of a running coupling constant. If we assume that the number of colours, N , is large the parameter $1/N$ becomes small and therefore can be used as an expansion parameter. 'tHooft [3] has shown that in this expansion the Yang-Mills Feynman graphs look very much like the string theory diagrams. To see this, it is convenient to adopt a double line notation in which any object in the adjoint representation of the gauge group $U(N)$ is represented by two lines with arrows on them. Thus the gauge field A is denoted by $A_b^a \rightleftharpoons$

Using this notation the ordinary Feynman graphs like



will look like triangulations of a 2-dimensional surface,



This mapping can be made more precise. To this end consider the Yang-Mills lagrangian

$$\mathcal{L} = \frac{1}{g_{\text{YM}}^2} \left[-\frac{1}{4} \text{Tr} F^2 + \dots \right] .$$

To apply the $\frac{1}{N}$ expansion we need to keep the 'tHooft coupling constant $\lambda = g_{\text{YM}}^2 N$ fixed as $N \mapsto \infty$. With this redefinition \mathcal{L} looks like

$$\mathcal{L} = \frac{N}{\lambda} \left[-\frac{1}{4} \text{Tr} F^2 + \dots \right] .$$

We can now read the N - and λ -dependence of each graph. We note the following sources of N -dependence in each graph: a factor of $\frac{\lambda}{N}$ coming from each propagator, a factor of $\frac{N}{\lambda}$ coming from each vertex, and a factor of N coming from each loop. Therefore, a general graph with h loops (faces), v vertices and p propagators will have the following N -dependence

$$N^h \left(\frac{\lambda}{N} \right)^p \left(\frac{N}{\lambda} \right)^v = N^{h-p+v} \lambda^{p-v} .$$

The power of N is the Euler number of a two dimensional surface which is defined by

$$\begin{aligned} \chi &= h - p + v \\ &= 2 - 2g \end{aligned}$$

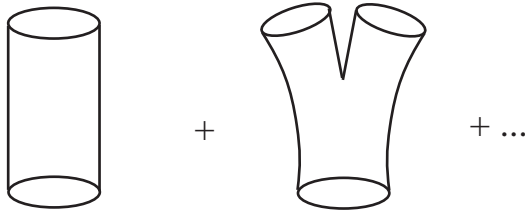
where g is the topological genus of the surface. If we identify $1/N$ with a string theory coupling constant the Yang-Mills perturbative expansion will go over to a topological expansion in powers of the genus of two dimensional surfaces which is characteristic for perturbative expansion in string theory. Note that the partition function of the Yang-Mills theory, which is obtained from the summation of all the Yang-Mills vacuum graphs, will have the general form of

$$\sum_{\substack{g=0 \\ h=1}} C_{g,h} N^{2-2g} \lambda^{2g-2+h}$$

where $C_{g,h}$ indicate the result of loop integration and other algebraic operations on each Feynman diagram in the Yang-Mills theory. This expression can be given both an open string as well as a closed string interpretations.

1) Open string interpretation

The above result for the gauge theory partition function indicates that with each gauge theory diagram with h loops, v vertices and p propagators is associated a factor of $N^h (g_{YM}^2)^{p-v}$. With an appropriate interpretation of the parameters this factor becomes identical to the one which one associates to an open string diagram on a world sheet with g handles and h boundaries.² At a more intuitive level it is seen that the open string diagrams with g handles and h boundaries like



can be mapped to the gauge theory Feynman graphs simply by flattening them,

² For the Chern-Simons theory we should replace g_{YM}^2 with $\frac{2\pi N}{N+k}$, where k is the Chern-Simons coupling constant.



A concrete realization of this qualitative picture has been given by Witten [10]. Witten considers a gauge theory of Chern-Simons type on S^3 with the partition function

$$Z = \int [DA] e^{\frac{k}{4\pi} \int_{S^3} \text{Tr}(AdA + A^3)}$$

and shows that the $C_{g,h}$ corresponding to this partition function equals the partition function of a topological open string theory on a world sheet with g handles and h boundaries. Moreover the target space of this string theory, which is nothing but the topological A model, is the cotangent bundle T^*S^3 of S^3 in a background of N D_3 branes wrapped on a S^3 submanifold. The cotangent bundle of S^3 is isomorphic to the manifold of the group $SL(2, C)$ which is defined by

$$T^*S^3 : XY - UV = 1$$

This manifold in turn is isomorphic to

$$XY - UV = \mu^2$$

which is a deformed conifold.

2) Closed string interpretation

Assume that we can perform the sum over h , i.e. number of holes on the open string world sheet. The partition function then reduces to a sum over g of the form

$$\sum_{g=0} N^{2-2g} F_g(\lambda) . \quad (1)$$

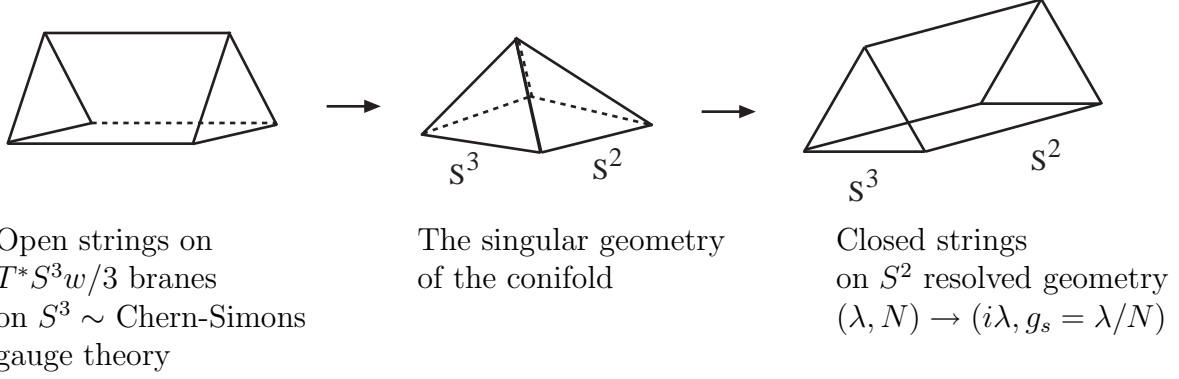
This can be interpreted as a closed string perturbative expansion with the string coupling constant g and the string (length)² α' identified as

$$\frac{1}{N} \sim g_s , \quad \lambda \sim \frac{1}{\alpha'} .$$

Note that $N \rightarrow \infty$ implies $g_s \rightarrow 0$, i.e. string weak coupling limit. This is the perturbative regime in string theory. Thus the large N limit of Chern-Simons theory maps to string perturbative regime, as argued above. It also becomes plausible to assume that summing over the number of boundaries of the open string theory produces closed surfaces of the closed string theory.

A concrete realization of such resummation has been performed recently by Gopakumar and Vafa [11]. These authors show that the partition function of the Chern-Simons theory on S^3 can indeed be mapped to a topological string propagating on a small resolution of the conifold.

Graphically it seems that we have the following type of transition



The Gopakumar-Vafa mapping starts from a detailed analysis of the Chern-Simons partition function on S^3 which is known to be

$$\begin{aligned}
 Z(S^3, N, h) &\equiv e^{-F(S^3, N, k)} = \\
 &= \frac{1}{(N+k)^{N/2}} \prod_{j=1}^{N-1} \left\{ 2 \sin \left(\frac{j\pi}{N+k} \right) \right\}^{N-j} .
 \end{aligned}$$

Expanding this expression in powers of $1/N$ produces

$$F = \sum_{g=0} N^{2-2g} F_g(\lambda)$$

where λ is defined by

$$\lambda = \frac{2\pi N}{N+k} .$$

One can now use the explicit form of $F(S^3, N, k)$ to calculate $F_g(\lambda)$. For example the $g=0$ term is given by

$$N^2 F_0(\lambda) = - \left(\frac{N^2}{\lambda} \right) \left[-\zeta(3) + \frac{i\pi^2}{6} \lambda - \left(m + \frac{1}{4} \right) \pi \lambda^2 + \frac{i\lambda^3}{12} + \sum_{n=1}^{\infty} \frac{e^{-in\lambda}}{n^3} \right] .$$

With the substitution of $g_s = \frac{i\lambda}{N}$ and $t = i\lambda$ this goes over to

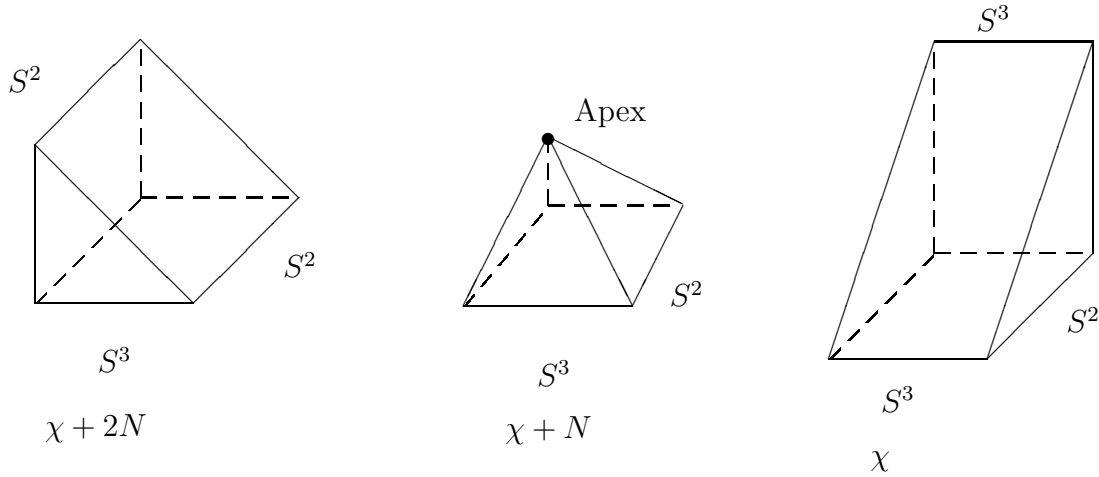
$$\frac{1}{g_s^2} \left[-\zeta(3) + \frac{\pi^2}{6} t + i \left(m + \frac{1}{4} \right) \pi t^2 - \frac{t^3}{12} + \sum_{n=1}^{\infty} \frac{e^{-nt}}{n^3} \right] .$$

This result agrees with $F_0(g_s, t)$ for a closed string on a S^2 resolved conifold. The $\sum_{n=1}^{\infty} \frac{e^{-nt}}{n^3}$ terms come from the world sheet instantons. Similar match have been

obtained by Gopakumar and Vafa for $g \geq 1$ partition functions as well. At least in the context of this simple setting it seems that a gauge theory of Chern-Simons type can be considered as a non perturbative version of gravity.

The closed and open string interpretation of the same Chern-Simons theory is rather similar to the duality in AdS/CFT correspondence which we are going to briefly discuss in the next paragraph. As Gopakumar and Vafa suggest one can think of S^2 as the two sphere surrounding the positions of the D_3 branes in a transverse R^3 subspace inside T^*S^3 .

Historically conifold entered string theory in connection with the Calabi-Yau compactification from ten to four dimensions. For example it has been observed by Candelas and de la Ossa that, in the context of string theory, it is possible to make transitions between Calabi-Yau threefolds of different topologies by passing through conifold points in the moduli space. Graphically one can represent this process by [6].



where N is the number of nodal points.

Starting from the smooth three-folds on the right of the above figure we can shrink N S^3 's to single points thereby ending up at a conifold point in the moduli space. Since the Euler number of each point is unity we end up with a singular three fold with an Euler number $\chi + N$, where χ is the Euler number of the initial 3-fold. Now we can replace each singular point with a S^2 and obtain a new smooth 3-fold with an Euler number $\chi + 2N$. This idea has been used by Strominger in 1995 in his famous work on conifold transition [7]. Strominger observed that the low energy physics of the moduli fields in the Calabi-Yau compactifications of type II theories is governed by some σ model targeted on the moduli space of the compactifying manifold. As these fields vary in time they encounter a conifold point thereby generating a singularity in the

low energy theory. Strominger then gave a very nice physical understanding of these singularities. For the sake of concreteness let us consider type IIB compactification and a D_3 brane wrapped around an S^3 in the compactifying Calabi-Yau three-fold. As the S^3 shrinks to zero size the 3-fold approaches a conifold singularity. It also generates a state which looks like a black hole from the 4-dimensional point of view. Because of the vanishing volume of the wrapped brane this black hole will be massless and will give rise to a massless multiplet in the low energy effective 4-dimensional field theory. The origin of the singularity is the appearance of this massless multiplet.

4. D_3 Branes and the AdS/CFT Correspondence ³

A relationship like the one outlined in the previous section between gauge theory and gravity in 4 dimensions would be highly interesting and will obviously lead to a deeper understanding of both gauge theory and gravitational physics. To make progress we need to impose some additional simplifying restrictions like supersymmetry, conformal symmetry or rather super-conformal symmetry. With these restrictions some concrete results have been obtained by Gubser, Klebanov, Polyakov[15], Maldacena [2] and Witten [15]. The starting point is the IIB string on $AdS_5 \times S^5$ background. These authors have argued that the $\mathcal{N} = 4$, $D=4$ super Yang-Mills on \mathbb{R}^4 is equivalent to IIB strings on $AdS_5 \times S^5$.

From the point of view of our discussion in previous paragraphs we can say that for $\mathcal{N} = 4$, $D = 4$ Yang-Mills theory the dual string theory having the partition function given in (1) is the type IIB superstring on the $AdS_5 \times S^5$ background. The background also needs to have a RR charge and a curvature related to λ . As we shall argue presently this duality will be valid in the weak coupling limit of the string theory.

To give some more details of the gauge theory-gravity correspondence in 4-dimensions we need to go a little more into the description of the D brane physics. The semiclassical D_3 brane solution, which is a BPS state, is obtained by setting to zero the supersymmetry variations of fermionic fields in the type IIB theory. The N parallel D_3 brane configuration is given by,

$$ds^2 = H^{-1/2}(r)[-dt^2 + dx^2] + H^{1/2}(r)[dr^2 + r^2 g_{ij} dy^i dy^j]$$

where

$$H(r) = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s N \alpha'^2$$

with N an integer and g_s the string coupling constant. The part $-dt^2 + dx^2$ defines a flat metric in \mathbb{R}^4 , while $dr^2 + r^2 g_{ij} dy^i dy^j$ in general defines only a Ricci flat metric in a 6-dimensional transverse space Y_6 . The Ricci flatness of Y_6 requires that the metric $g_{ij} dy^i dy^j$ on the 5 manifold X_5 spanned by the coordinates y^i is Einstein with

³For a brief review see [12]. For longer reviews see [13].

a cosmological constant equal to 4. Unless $X_5 = S^5$ the 6-manifold Y_6 is singular at $r = 0$. In the singular case Y_6 is called a conifold because it admits a group of diffeomorphisms, $r \rightarrow tr$ with $t > 0$. Y_6 is a cone over X_5 . The better known D_3 brane solution corresponds to $Y_6 = \mathbb{R}^6$ and $X_5 = S^5$.

Note that as $r \rightarrow \infty$ the function H approaches 1 and the manifold splits into $\mathbb{R}^4 \times Y_6$. For $Y_6 = \mathbb{R}^6$ we obtain the 10-dimensional Minkowski vacuum. As $r \rightarrow 0$ we obtain $AdS_5 \times X_5$. If $Y_6 = \mathbb{R}^6$ we obtain the near horizon geometry $AdS_5 \times S^5$. We thus see that the brane configuration extrapolates between the vacuum Kaluza-Klein type solutions at $r = 0$ and $r = \infty$.

The number of the unbroken supersymmetries depends on the choice of Y_6 . For $Y_6 = \mathbb{R}^6$ only 16 of the original 32 real supersymmetries will leave the brane configuration invariant. For other choices of Y_6 the number of the unbroken supersymmetries will be smaller. One important feature, however, is that as we approach the limiting geometries at $r = 0$ or ∞ both the bosonic and fermionic symmetries of the configuration increase. For arbitrary values of r the bosonic symmetries are the 4-dimensional 10 parameter Poincaré group times a subgroup of the isometry group of Y_6 which leaves the function $H(r)$ invariant. For the case of $Y_6 = \mathbb{R}^6$ this is $SO(6)$. As we approach $r = 0$ or ∞ the symmetry enlarges to $SO(2, 4) \times SO(6)$ (near $r = 0$) or the 55 parameter Poincaré symmetries of \mathbb{R}^{10} (near $r = \infty$). In this case the number of supersymmetries also increase to 32 in the two extreme limits.

The $r \rightarrow 0$ region always has the AdS_5 as part of its geometry. Therefore, considering $AdS_5 \times X_5$ as a Kaluza-Klein type solution of type IIB supergravity the effective low energy theory will have the group $SO(2, 4)$ as part of its symmetry group. $SO(2, 4)$ is not only the isometry group of AdS_5 it is also the group of conformal transformations of \mathbb{R}^4 . The relevance of \mathbb{R}^4 to this discussion is due to the fact that it appears as the boundary of AdS_5 . Along with $SO(2, 4)$, the isometry group of X_5 as well as the unbroken supersymmetries will be contained in the total symmetry group of the Kaluza Klein background $AdS_5 \times X_5$.

For the case of $X_5 = S^5$ the low energy perturbative description of N parallel D_3 branes is in terms of $\mathcal{N} = 4$, $U(N)$ supersymmetric Yang-Mills theory in \mathbb{R}^4 . In this case the total symmetry group is $SU(2, 2|4)$.

For the case of $Y_6 = \mathbb{R}^6$, the AdS/CFT correspondence asserts that, within a certain region of the parameter space, one can obtain nonperturbative information about the $\mathcal{N} = 4$, $SU(N)$ Yang Mills theory from the low energy type IIB supergravity on the $AdS_5 \times S^5$ background. For this description to be a good approximation the radius R of S^5 must be large with respect to the $d = 10$ Planck length. From the above explicit formula for R it is seen that a large value of R^2 (with respect to α') requires a large value for $g_s N$. Since we are ignoring string loop effects we should demand that g_s is small. We will thus be dealing with a large N gauge theory in 4 dimensions. A meaningful formulation of large N gauge theories in $d = 4$ requires that the Yang-Mills coupling constant $g_{YM}^2 = g_s$ must decrease as we increase N such that the 'tHooft coupling $\lambda = g_s N$ remains fixed. In our problem λ has to be

large. We thus end up in the regime of strong 'tHooft coupling limit of the $d = 4$ super Yang-Mills theory, a domain in which the standard perturbative methods are not applicable.

This strong-weak duality also explains the discrepancy in the results of the entropy calculations in Yang-Mills and string descriptions which were puzzling when they were first calculated. Here is a brief description of these results. The black hole entropy of a near extremal 3-brane of Hawking temperature T turns out to be [17]

$$S_{BH} = \frac{\pi^2}{2} N^2 V_3 T^3$$

where N is interpreted as the charge of the brane and V_3 is the spatial volume of the brane. On the other hand the entropy of a free gas of particles in the $D = 4$, $\mathcal{N} = 4$ multiplet of the $U(N)$ gauge theory turns out to be

$$S = \frac{2\pi^2}{3} N^2 V_3 T^3 .$$

Note that the $V_3 T^3$ behaviour is dictated by conformality of the 4-dimensional $\mathcal{N} = 4$ super Yang Mills theory. The appearance of N^2 is a reflection of degrees of freedom of $U(N)$, as all the fields are in the adjoint representation of this group. It is thus seen that apart from a relative factor of $3/4$ the two entropies are the same. This discrepancy, which created some confusion when it was first observed, is in fact welcome, because, we now know that the Yang Mills entropy is obtained through a weak coupling perturbative calculation. According to what we said above it is the limit of the infinite 'tHooft coupling at which we expect to obtain agreements with the supergravity calculations. It has been shown that in fact the factor of $3/4$ connects the two limits of the theory [18].

At the level of matching the spectra of the Kaluza-Klein theory on the maximally symmetric $AdS_5 \times S^5$ background with that of the $\mathcal{N} = 4$ gauge theory on \mathbb{R}^4 the AdS/CFT conjecture makes perfect sense. It essentially is a restatement of the symmetries of the two theories [14]. The correspondence is less obvious when the background has fewer symmetries. It is this less symmetric case which will be the focus of our main interest in the following paragraphs.

In the maximally symmetric case every Kaluza-Klein mode on S^5 is dual to an operator in the Yang Mills theory. Let J be one such mode and \mathcal{O} its dual operator on the Yang-Mills side. A very explicit formulation of the correspondence states that the generating functional for the correlation functions of \mathcal{O} can be obtained by evaluating the type IIB supergravity partition function on the $AdS_5 \times S^5$ background subject to the boundary condition that as we approach the boundary of AdS_5 the mode J approaches a boundary field \hat{J} [15]. The leading order contribution to the generating functional of the connected Green's functions on the Yang-Mills side will be given by the value of the classical action of type IIB theory evaluated at a particular solution of equations of motion subject to the given boundary conditions.

From the brief sketch presented in the foregoing sections it becomes apparent that the strong coupling limit of the Yang-Mills theory in $D = 4$ needs new degrees of freedom. Whereas the standard perturbative analyses are adequate for weak couplings, the strong coupling domain of the same gauge theory simplifies in the dual string description. From these analyses it becomes also understandable why the attempts over the years to derive a stringy behaviour from the dynamics of Yang-Mills theory have failed. The reason obviously is that the gravitational degrees of freedom were missing. Maldacena's conjecture shows that it is the throat region ($r \ll R$) of the D_3 brane geometry in $D = 3$ dimensions which should give a meaningful description of the strongly coupled gauge theory in four space time dimensions.

5. D_3 Branes Near Conifolds

One of the choices of Y_6 which leads to a less symmetric configuration is the surface

$$Z_1 Z_4 - Z_2 Z_3 = 0$$

defined in \mathbb{C}^4 . By a simple change of coordinates this surface can also be written as

$$\zeta_1^2 + \dots + \zeta_4^2 = 0$$

which is the cone we encountered before. As we said earlier its base X_5 is obtained by intersecting the surface with the sphere

$$|\zeta_1|^2 + \dots + |\zeta_4|^2 = 1$$

We can rotate the four variables ζ^i by elements of $O(4)$. This group acts transitively on the above intersection which defines the base of the cone. The base X_5 , therefore, has to be a coset space. To identify it consider some point, for example $(\frac{1}{\sqrt{2}}, 0, 0, \frac{i}{\sqrt{2}})$ and evaluate its stability subgroup, which is obviously the $O(2)$ subgroup of $O(4)$ acting on the 2 – 3 plane. We thus have $X_5 = SO(4)/SO(2)$. $SO(4)$ invariant metrics on these manifolds have been worked in [6]. The metrics in general depend on two parameters n and n' . It is customary to denote the corresponding manifolds by $T^{nn'}$. The explicit form of the metric on $T^{nn'}$ is,

$$ds^4 = c^2(dy_5 - n \cos y_1 dy_2 - n' \cos y_3 dy_4)^2 + a^2(dy_1^2 + \sin^2 y_1 dy_2^2) + a'^2(dy_3^2 + \sin^2 y_3 dy_4^2),$$

All the coordinates y^α are angles, such that $y = (y^1, y^2)$ parametrize a S^2 of radius a while $y' = (y^3, y^4)$ parametrize a S^2 of radius a' . The angle y^5 ranges from 0 to 4π . The constant c is the radius of the circle defined by y^5 . The constants n and n' will be taken to be integers. Locally the manifold looks like $S^2 \times S^2 \times S^1$. Globally $T^{nn'}$ is a $U(1)$ bundle over $S^2 \times S^2$. The isometry group is $SU(2) \times SU(2) \times U(1)$, where the $U(1)$ factor is due to the translational invariance of the coordinate y^5 .

In order for $AdS_5 \times T^{nn'}$ to be a supersymmetric solution of the type IIB field equations it is necessary that $a = a' = \frac{1}{\sqrt{6|e|}}$, $n = \pm n'$ and $ec = -\frac{1}{3n}$, where, e^2 is related to the AdS_5 cosmological constant through $R_{\mu\nu} = -4e^2 g_{\mu\nu}$. It has been argued by Klebanov and Witten that the $U(1)$ factor should be identified with the R symmetry of the world volume $\mathcal{N} = 1$, $d = 4$ superconformal field theory which arises as a consequence of the AdS/CFT correspondence [5]. It has been argued in [16] that there is a possible connection of the present theory with the $c = 1$ non critical string theory. In this context it becomes plausible that the R charge is also put in correspondence with the $U(1)$ group generated by the Liouville mode of this theory. In this way the R charges of the boundary $d = 4$ superconformal theory will be set in correspondence with the Liouville momenta of the $c = 1$ theory.

We know from the perturbative description of a collection of N coinciding D_3 branes that the low energy description should be in terms of a super symmetric Yang-Mills theory on the world volume of the brane. What we are arriving at is that in the limit of the strong 'tHooft coupling the type IIB superstring on $AdS_5 \times T^{1,1}$ is dual to the large N limit of a super Yang-Mills theory on the world volume of the brane. In fact Klebanov and Witten [4] have shown that the correct super Yang-Mills theory entering this duality is $\mathcal{N} = 1$ superconformal $SU(N) \times SU(N)$ gauge theory on $\mathbb{R}^{1,3}$, at a very particular fixed point which we are going to outline presently. Before doing this we note that the background we are considering has a global symmetry of $SU(2)_j \times SU(2)_\ell \times U(1)_R$. The Yang-Mills theory has scalar chiral superfields A_i , B_i and vector multiplets W_α^1 and W_α^2 which belong to the following representations of $SU(N) \times SU(N) \times SU(2)_j \times SU(2)_\ell \times U(1)_R$

$$\begin{aligned} A_i &\sim (N, \bar{N}; 2, 1)_{1/2} & B_i &\sim (\bar{N}, N; 1, 2)_{1/2} \\ W_\alpha^1 &\sim (N^2 - 1, 1; 1, 1)_1 & W_\alpha^2 &\sim (1, N^2 - 1; 1, 1)_1 \end{aligned}$$

Klebanov and Witten associate the following conformal dimensions with these operators

$$\begin{aligned} \Delta(A) &= \Delta(B) = \frac{3}{4} \\ \Delta(W) &= \frac{3}{2} \end{aligned}$$

Using the above super fields one can construct gauge invariant chiral operators. Here are some examples,

$$\begin{aligned} \text{Tr}(AB)^k & \quad \Delta = \frac{3}{2}k & R = k \\ \text{Tr}(W_\alpha(AB)^k) & \quad \Delta = \frac{3}{2}(k+1) & R = k+1 \\ \text{Tr}(W^\alpha W_\alpha(AB)^k) & \quad \Delta = 3 + \frac{3}{2}k & R = k+2 \end{aligned}$$

For a more complete list see A. Ceresole et al. [19]. The question is how does one generate this spectrum of dimensions and R charges on the supergravity side. The AdS/CFT correspondence states that the conformal dimensions of the chiral operators on the Yang-Mills side are given in terms of the masses of the Kaluza-Klein modes. For example for a gauge invariant Yang-Mills operator dual to a p -form the relation is as follows:

$$\Delta = 2 + \sqrt{(4-p)^2 + 4(mR)^2}$$

where m = Kaluza-Klein mass of a mode originating from a p -form field on X_5 and it has the general form of (at least for the case of $X_5 = S^5$) $m = \frac{n}{R}$, with $n \in \mathbb{Z}$.

The dimensions are thus independent of R and therefore they are also independent of the 'tHooft coupling. Likewise a gauge invariant Yang-Mills operator dual to a massive spin 3/2 field on the supergravity side the formula is

$$\Delta = 2 + |m + 3/2|$$

It becomes therefore important to find the masses of the Kaluza-Klein modes. This problem has been solved completely [16],[19]. To perform the spectral analysis one notices that $T^{1,1}$ is a magnetic monopole bundle over $S^2 \times S^2$. So if we expand on the fibre coordinate y_5 we obtain fields defined on $S^2 \times S^2$, viz,

$$\Phi(y_1, y_2; y_3, y_4, y_5) = \sum_{s \in \frac{1}{2}\mathbb{Z}} \Phi_s(y_1, y_2, y_3, y_4) e^{is y_5}$$

Φ_s are coupled to spin connections on $S^2 \times S^2$ as well as to magnetic monopole fields

$$A = \cos y_1 dy_2 \quad A' = \cos y_3 dy_4$$

We can expand Φ_s on the basis of Wigner functions on $S^2 \times S^2$. The spectrum is classified according to $SU(2)_\ell \times SU(2)_j \times U(1)_s$ representations $(\ell, j; s)$. Following procedures developed in [20], one can obtain the eigenvalues of various laplacian type operators on $T^{1,1}$. For example the scalar Laplacian has the following eigenvalues

$$H(s) = 6 \left(\ell(\ell+1) + j(j+1) - \frac{s^2}{2} \right)$$

$$\ell \geq |s| \quad j \geq |s|$$

Likewise for the Dirac operator \not{D} we obtain

$$\frac{1}{2} \pm \sqrt{H\left(s - \frac{1}{2}\right) + 4}$$

$$-\frac{1}{2} \pm \sqrt{H\left(s + \frac{1}{2}\right) + 4}$$

These eigenvalues give the masses of various AdS fields. It must be noted that, since the eigenvalues of various Laplacians are irrational functions of the quantum numbers, the masses will also be irrational functions of ℓ , j and s . Therefore the conformal dimensions Δ which are equal to the *AdS* energy become irrational functions of ℓ , j and s ,

$$\begin{aligned}\Delta(\psi_\mu) &= 2 + |m_{s/2} + 3/2| \\ \Delta(\lambda) &= 2 \pm |m_{1/2}|\end{aligned}$$

We thus see that the boundary conformal field theory contains infinite number of operators with irrational dimensions [21]. However, most of the modes at the bottom of Kaluza-Klein towers have rational dimensions. For example for the scalars such modes have the quantum numbers $j = \ell = s$ and masses

$$m^2 = H(s) = (3s + 2)^2 - 4$$

Their conformal dimensions will therefore be given by

$$\begin{aligned}\Delta_\pm &= 2 \pm \sqrt{m^2 + 4} \\ &= 2 \pm (3s + 2)\end{aligned}$$

These correspond to the chiral operators $Tr(AB)^k$ on the Yang-Mills side

$$\Delta = \frac{3k}{2} \quad \ell = j = \frac{k}{2}$$

For this particular example it is also instructive to work out the corresponding supergravity modes. To this end one writes the metric and the 4-form potential on $T^{(1,1)}$ as

$$\begin{aligned}g_{\alpha\beta} &= \bar{g}_{\alpha\beta} + h_{\alpha\beta} \\ A_{\alpha_1 \dots \alpha_4} &= \bar{A}_{\alpha_1 \dots \alpha_4} + a_{\alpha_1 \dots \alpha_4}\end{aligned}$$

where \bar{g} and \bar{A} are background fields. The $D = 10$ supergravity equations for these perturbations leads to a 2×2 sector involving h^α_α and $a_{\alpha_1 \dots \alpha_4}$. Diagonalizing the linearized field equations one obtains the masses

$$\begin{aligned}m_\pm^2 + 4 &= \left(\sqrt{H + 4} \pm 4\right)^2 \\ &= ((3s + 2) \pm 4)^2 \\ j &= \ell = s\end{aligned}$$

Substitute m_-^2 in Δ_+

$$\begin{aligned}\Delta_+ &= 2 + \sqrt{m_-^2 + 4} \\ &= 3s = 3\frac{k}{2} \quad \text{if } s = \frac{k}{2}\end{aligned}$$

Many other states with rational dimensions have been matched in both sides. They all fall into *short* multiplets of $SU(2, 2|1)$. Thus their dimensions are protected.

To summarize conifolds appear in many places in string theory like the $c = 1$ non-critical strings, Chern Simons theory on S^3 , compactification of IIB and D_3 -branes in IIB etc. In the case of D_3 -branes they lead to $\mathcal{N} = 1$ superconformal $SU(N) \times SU(N)$ Yang-Mills on the boundary of AdS_5 . Many Kaluza-Klein modes have been matched with the *short* multiplets on the YM side. There are infinite numbers of Kaluza Klein towers which lead to irrational dimensions on the Yang-Mills side. This is unlike the $AdS_5 \times S_5$ case. The symmetry group is

$$SU(2, 2|1) \times SO(4)$$

One may wonder if the $c = 1$ non-critical string theory plays a role in the boundary Yang Mills theory?

6. Relation to $c=1$ string theory

The fundamental fields of the $c = 1$ theory are two scalars X and ϕ and the b, c ghost fields. X is targeted on a S^1 while ϕ is coupled to a background charge. At the self dual radius $R = 1/\sqrt{2}$ of the circle there is a $SU(2) \times SU(2)$ symmetry. The BRS cohomology classes are organized according to the representations of this group. These classes are also labelled by their ghost numbers. Of interest to us are the ghost number zero, one and two operators given respectively by $\mathcal{O}_{s,p}(z)\overline{\mathcal{O}}_{s,p'}(\bar{z})$, $Y_{s+1,p}^+(z)\overline{\mathcal{O}}_{s,p'}(\bar{z})$ as well as $a(z, \bar{z})\mathcal{O}_{s,p}(z)\overline{\mathcal{O}}_{s,p'}(\bar{z})$ and $Y_{s,p}^+(z)\bar{Y}_{s,p'}^+(z)$. Here we follow the notation of [9]. The complex conjugates of these operators should also be added to the list. In each case the subscript s characterizes the $SU(2) \times SU(2)$ content of each object. For a given integer or $1/2$ integer s the indices p and p' range from $-s$ to $+s$.

Now consider a Kaluza-Klein tower originating from a q -form field in T^{11} . For a given $U(1)$ charge s we consider the modes at the bottom of each tower (those which presumably have rational conformal dimensions in the boundary gauge theory). An observation made in [16] is that these Kaluza-Klein modes are in correspondence with the ghost number q cohomology classes in the $c = 1$ theory. Note that the modes originating from the components of the metric in T^{11} (which are not q -forms in the internal space!) do not seem to have a counterpart in the $c = 1$ side. Furthermore the modes corresponding to an operator containing $a(z, \bar{z})$ in the $c = 1$ side can actually be gauged away in the Kaluza-Klein side.

The $c = 1$ theory has an infinite dimensional algebra given in terms of the volume preserving diffeomorphisms of the quadric cone $Z_1 Z_2 - Z_3 Z_4 = 0$. As we mentioned in previous section the base of this cone is isomorphic to T^{11} . In the context of present discussion the 3 complex dimensional Ricci flat cone is in fact identical to the subspace transverse to the D_3 -brane solution of the type IIB supergravity. Our

Kaluza-Klein background is a near horizon approximation to this D_3 brane geometry. Thus the cone seems to be the common geometrical entity in the two very different looking theories⁴.

The cone is singular at its apex and as we discussed in the previous section one can resolve the singularity by deforming the defining equation into $Z_1 Z_2 - Z_3 Z_4 = \mu$. From the point of $c = 1$ theory μ corresponds to the 2-dimensional cosmological constant. One can also consider a topological σ model targeted on a CY three fold near a conical singularity, for which the local equation is the same as our quadratic expression. For both of these theories the free energies can be evaluated as a function of μ and can be expressed as a genus expansion.

In [8] Ghoshal and Vafa argued that in fact the two theories must be the same. They observed that, at the self dual radius, the $g = 0, 1$ and 2 contributions to the free energy of the $c = 1$ theory agree with the corresponding terms of the free energy of the topological sigma model near the conifold singularity. Subsequently, assuming the type II-heterotic duality, the results of [22] gave further support to the Ghoshal-Vafa conjecture. These authors calculated the coefficient of the term $R^2 F^{2g-2}$ in the effective action of the heterotic theory compactified on $K_3 \times T^2$ and realized that for any g the coefficient is also given by the genus g term of the partition function of the $c = 1$ theory at the self dual radius. More recently Gopakumar and Vafa [23] have calculated the σ -model partition function near a conifold singularity and have proven the conjecture made in [8].

A better understanding of the correspondences noted above may require the unraveling of the relevance of the volume preserving diffeomorphisms of the cone in the D_3 brane context. On the basis of the observations made in this note we would like to think that the $c = 1$ theory at the self dual radius has a role to play in organizing the chiral primaries of the boundary $SU(N) \times SU(N)$ superconformal gauge theory.

Acknowledgements

I would like to thank Matthias Blau for useful discussions. The very kind hospitality of the organizers of the Conference on Quantum Symmetries in Goslar and Wigner Symposium in Istanbul is gratefully acknowledged.

⁴Similar remarks can be made about the manifolds T^{nn} . In this case we should consider the $c = 1$ string theory at n times the self-dual radius.

References

- [1] J. Polchinski, *String Theory*, Vol 1 and 2, CUP (1988).
- [2] J. Maldacena, “*The large N limit of superconformal field theories and supergravity*”, hep-th/9711200.
- [3] G. t’Hooft “*A planar diagram theory for string interactions*”, Nucl. Phys. **B72** (1974) 461.
- [4] D. Gross, Nucl. Phys. **B400** (1993) 161; D. Gross and W. Taylor, Nucl. Phys. **B400** (1993) 181; D. Gross, Nucl. Phys. **B403** (1993) 305.
- [5] I.R. Klebanov and E. Witten, “*Superconformal field theory on three-branes at a Calabi-Yau singularity*”, hep-th/9807080.
- [6] P. Candelas and X. de la Ossa, “*Comments on conifolds*”, Nucl. Phys. **B342** (1990) 246.
- [7] A. Strominger, “*Massless black holes and conifolds in string theory*”, hep-th/9504090.
- [8] D. Ghoshal and C. Vafa, “ *$c = 1$ string as the topological theory in the conifold*”, Nucl. Phys. **B453** (1995) 121.
- [9] E. Witten, “*Ground ring of two dimensional string theory*”, hep-th/9108004; E. Witten and B. Zwiebach, “*Algebraic structures and differential geometry in 2D string theory*”, hep-th/9201056.
- [10] E. Witten, “*Chern-Simons theory as a strong theory*”, hep-th/9207094.
- [11] R. Gopakumar and C. Vafa, “*On the gauge theory/geometry correspondence*”, hep-th/9811131.
- [12] Michael R. Douglas and S. Randjbar-Daemi, “*Two lectures on AdS/CFT correspondence*”, hep-th/9902022.
- [13] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N gauge theories, string theory and gravity*, hep-th/9905111; J.L. Peterson, *Introduction to the Maldacena conjecture on AdS/CFT*, hep-th/9902131; P. Di Vecchia, “*Large N gauge theories and AdS/CFT correspondence*”, hep-th/9908148; E.T. Akhmedov, “*Introduction to the AdS/CFT correspondence*”, hep-th/9911095.
- [14] L. Andrianopoli and S. Ferrara, “*K-K excitations on $AdS_5 \times S_5$ as $N = 4$ ‘primary’ superfields*”, hep-th/9803171.

- [15] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “*Gauge theory correlators from non-critical string theory*”, hep-th/9802109; E. Witten, “*Anti-de Sitter space and holography*”, hep-th/9802150.
- [16] D. Jatkar and S. Randjbar-Daemi, “*Type IIB string on $AdS_5 \times T^{nn'}$* ”, hep-th/9904187.
- [17] S.S. Gubser, I.R. Klebanov and A.W. Peet, “*Entropy and temperature of black 3-branes*”, hep-th/9602135.
- [18] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, “*Coupling constant dependence in the thermodynamics of $N = 4$ super Yang-Mills theory*”, hep-th/9805156.
- [19] A. Ceresole, G. Dall’Agata, R. D’Auria and S. Ferrara, “*Spectrum of type IIB supergravity on $AdS_5 \times T^{11}$: Predictions on $N = 1$ SCFT’S*”, hep-th/9905226; A. Ceresole, G. Dall’Agata and R. D’Auria, “*KK spectroscopy of type IIB on $AdS_5 \times T^{11}$* ”, hep-th/9907216.
- [20] S. Randjbar-Daemi, A. Salam and J. Strathdee, “*Spontaneous compactification in six dimensional Einstein-Maxwell theory*”, Nucl. Phys. **B214** (1983) 491.
- [21] S. S. Gubser, “*Einstein manifolds and conformal field theories*”, hep-th/9807164.
- [22] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor “ *$N=2$ type II-heterotic duality and the higher derivative F -terms*”, hep-th/9507115.
- [23] R. Gopakumar and C. Vafa, “*M-theory and topological strings I and II*”, hep-th/9809187 and hep-th/9811131